Counting Bimonotone Subdivisions

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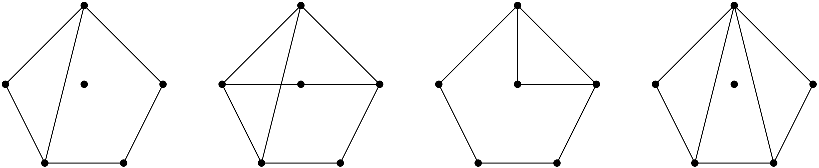

Subdivision | Not a subdivision | Not a subdivision | Triangulation

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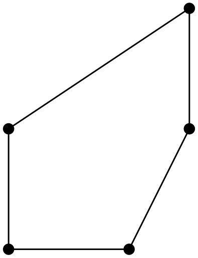
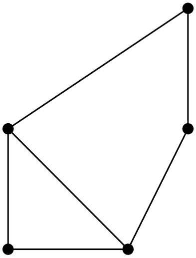


Bimonotone



Not bimonotone

Tent Functions

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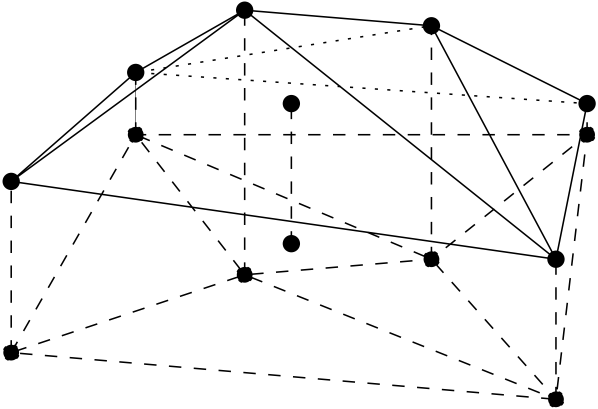

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- If f is supermodular, then the random variables defined by p are positively dependent on each other
- Example: An IQ test with n questions
    - The joint distribution of n scores takes $f(x)$
    - The score for each question has a density
    - Scores on separate questions are positively correlated

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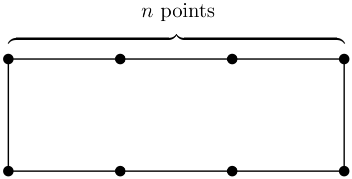
- For a tent function f, the subdivision is bimonotone if and only if f is supermodular
- The goal of this project is to count the number of bimonotone subdivisions and compare this to the total number of subdivisions

Our Work: $2 \times n$ Grids



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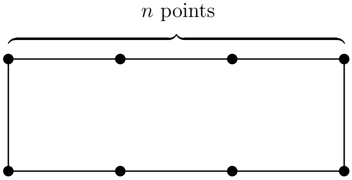
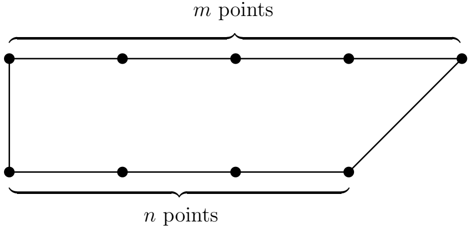


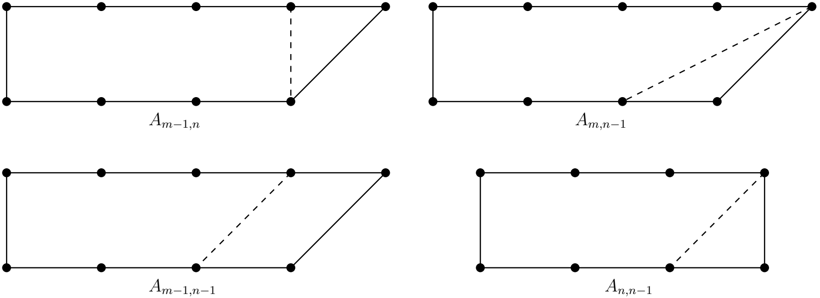
- First consider subdivisions of a $2 \times n$ lattice grid
- To use a recursion, we extend this to grids with m points at the top and n at the bottom



Recursion

- Using inclusion-exclusion for the unconnectedness of the top right and bottom right vertices, the number of bimonotone subdivisions is

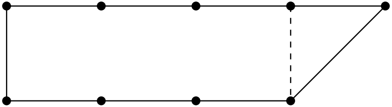
$$A_{m,n} = \begin{cases} 2A_{m,n-1} + 2A_{m-1,n} - 2A_{m-1,n-1}, & m > n \\ 2A_{m,n-1}, & m = n \\ 0, & m < n \end{cases}$$



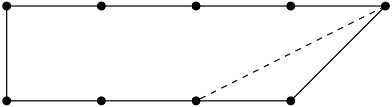
Recursion

- Similarly, for the total number of subdivisions,

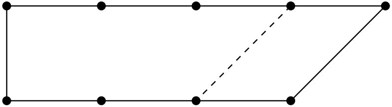
$$B_{m,n} = 2A_{m,n-1} + 2A_{m-1,n} - 2A_{m-1,n-1}$$



$B_{m-1,n}$



$B_{m,n-1}$



$B_{m-1,n-1}$

Theorem

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For a lattice grid with m points at the top and n points at the bottom:

- *The number of bimonotone subdivisions is given by $A_{m,n} = \frac{2^{m-2}}{(n-1)!} P_n(m)$, where $P_n(m)$ is some monic polynomial with degree $n-1$.*
- *The total number of subdivisions is given by $B_{m,n} = \frac{2^{m-2}}{(n-1)!} Q_n(m)$, where $Q_n(m)$ is some monic polynomial of degree $n-1$.*

Proof Idea

- Proof by induction
- We repeatedly substitute smaller terms into the recursion, giving for $A_{m,n}$:

$$\frac{2^{m-2}}{(n-2)!}\left(P_{n-1}(m) + \left(P_{n-1}(m) + P_{n-1}(m-1) + \cdots + P_{n-1}(n)\right)\right)$$

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- We find the highest degree term using Faulhaber's formula for the sum of the pth powers of the first m positive integers:

$$\sum_{k=1}^{m} k^p = \frac{m^{p+1}}{p+1} + \frac{1}{2}m^p + \sum_{k=2}^{p} \frac{B_k}{k!}\frac{p!}{(p-k+1)!}m^{p-k+1}$$

where the B_k are the Bernoulli numbers

Future Research

- Prove these conjectures:
  - The number of bimonotone subdivisions of a $2 \times n$ lattice grid is 2^{n-1} times the nth large Schröder number
  - The total number of subdivisions of a $2 \times n$ lattice grid is 2^{n-1} times the nth Delannoy number

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  - The total number of subdivisions of a $2 \times n$ lattice grid is 2^{n-1} times the nth Delannoy number
- Find recursive formulas for $3 \times n$ and larger lattice grids
- Find closed form expressions for the number of bimonotone/total subdivisions
- Extend formulas into higher dimensions

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